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Non-linear Sigma Models on a Half Plane

M. F. Mourad ¹ and R. Sasaki ²

*Yukawa Institute for Theoretical Physics,
Kyoto University,
Kyoto 606-01, Japan.*

Abstract

In the context of integrable field theory with boundary, the integrable non-linear sigma models in two dimensions, for example, the $O(N)$, the principal chiral, the CP^{N-1} and the complex Grassmannian sigma models are discussed on a half plane. In contrast to the well known cases of sine-Gordon, non-linear Schrödinger and affine Toda field theories, these non-linear sigma models in two dimensions are not classically integrable if restricted on a half plane. It is shown that the infinite set of non-local charges characterising the integrability on the whole plane is not conserved for the free (Neumann) boundary condition. If we require that these non-local charges to be conserved, then the solutions become trivial.

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1 Introduction

Non-linear sigma models in $1 + 1$ dimensions or two dimensional euclidean space have been discussed in various contexts in particle physics: as a theoretical laboratory for four dimensional gauge theories [1], in the reduction of string theory, as a model field theory with special geometrical features [2], etc. The non-linear sigma models in $2 (1 + 1)$ dimensions to be discussed in this paper are *harmonic maps* from a 2 dimensional space(-time) M into a target space TG . In other words, they are “free field theories” and the Lagrangians consist of the kinetic terms only. The non-linear structure of the target manifolds is the origin of the nontrivial and non-linear interactions. It is well known that when the target space TG is a group manifold or a riemannian symmetric space [3], the equation of motion can be expressed in a Lax pair form and that the existence of an infinite set of continuity equations is guaranteed [4, 5].

The integrable structures and *classical solutions* of various non-linear sigma models have been investigated for more than a decade. In particular, if the target space (TG) is a complex projective space (\mathbb{CP}^{N-1}) or complex Grassmannian manifolds, very simple classes of solutions, the *instantons* and *anti-instantons* [6] are obtained by assuming the *finite action*. They are simply obtained from the two-dimensional (anti) holomorphic functions, just like the ordinary two dimensional harmonic functions are obtained as the real and/or imaginary parts of the (anti) holomorphic functions. In the language of harmonic maps they are called the (anti) holomorphic maps. Due to the conformal invariance of the theory, the finite action solutions can be considered as the solutions on the sphere ($M = S^2$), which is obtained from the two dimensional euclidean space by the addition of the point at infinity (one point compactification). Later a very general class of non-holomorphic solutions on S^2 for the $O(N)$ [7] and \mathbb{CP}^{N-1} sigma models [8] and the complex Grassmannian sigma models [9] are obtained by a simple algebraic method, the Gram-Schmidt orthonormalisation. See also [10] for the corresponding

solutions of the “supersymmetric” Grassmannian sigma models.

The exact and factorisable *quantum S-matrices* for various non-linear sigma models have been known also for more than ten years [11]. They are obtained as solutions of Yang-Baxter equations with certain prescribed symmetries. In contrast the exact S-matrices of affine Toda field theories [12, 13, 14] are diagonal and thus they are trivial solutions of the Yang-Baxter equation.

In the present paper we address the problem of the *non-linear sigma models on a half plane*. *Can they retain the solvability if some suitable boundary conditions are imposed?*

From the field theoretical point of view this problem can be considered in the context of “integrable field theory on a half line” instead of those on the whole line. Here the central issue is whether the infinite set (or its suitable subset) of *conserved quantities* in involution, which guarantees the integrability, can be preserved or not on the half plane. Since the infinite set of *continuity equations* are a consequence of the equation of motion, which is independent of the boundary. It is the *boundary conditions* that determine if the conserved quantities are preserved or not.

For sine-Gordon [15, 16], non-linear Schrödinger [15] and affine Toda field theories [17], the integrable boundary conditions have been determined. Thus for these theories, the main objects of research is the effects of the boundary or the boundary conditions which replace the “asymptotic conditions” in field theory on the whole line. It should be remarked that in all the above mentioned theories the *free boundary condition*³, i.e., zero boundary potential, is always integrable.

From the algebraic point of view this problem is interesting in connection with certain extension of the Yang-Baxter equations and their related algebras. Namely the non-linear sigma models on a half plane might offer

³In the linear theory it is called the Neumann boundary condition $\frac{\partial u}{\partial x_2} = 0$ at the boundary, $x_2 = 0$.

very interesting examples of *factorisable scatterings with boundary* [18]. In this approach, it is assumed that when a particle hits the boundary it is reflected elastically (up to rearrangements among mass degenerate particles). The compatibility of the reflections and the scatterings constitutes the main algebraic condition, called Reflection equation [18, 16, 19], which extends the Yang-Baxter equation and the related algebras [20].

In this paper we concentrate on the aspects of the classical (or field theoretical) integrability of the problem. It is shown that the non-linear sigma model on a half plane with *free boundary* fail to preserve the infinite set of non-local charges which underlie the classical integrability on the whole plane. In other words, the asymptotic conditions in field theory cannot always be replaced by boundary conditions at finite points ⁴ within the context of integrable field theory.

This paper is organised as follows. In section 2 we start with the two dimensional harmonic functions which are a simplest and well known example of harmonic maps. Then the general two dimensional sigma models with an arbitrary target space TG is defined. The boundary conditions and other necessary notions are introduced here. In section three we briefly review the derivation of the infinite set of continuity equations after Brézin et al. [5]. Section 4 gives the main result in the abstract form. It is shown that for the non-linear sigma model with free boundary the conservation of the infinite set of non-local charges is not satisfied unless we additionally require that the first and second non-local charges to be conserved. This in turn result in an infinite set of conditions on the higher currents at the boundary. In sections 5, 6 and 7 we demonstrate the above result for the explicit models, $O(N)$, the principal chiral and CP^{N-1} and the complex Grassmannian sigma models. It is shown that the requirement of the conservation of the second non-local charge can only be met by *trivial solutions* among the real analytic

⁴Note that the periodic boundary conditions on a finite interval always preserves the classical integrability.

solutions. Section 8 is devoted to summary and comments.

2 Harmonic Maps

Let us start with the harmonic function on a half plane:

$$\begin{aligned} -\infty < x_1 < \infty \\ 0 \leq x_2 < \infty, \end{aligned} \tag{2.1}$$

as the simplest example of the harmonic map ⁵. It is described by the action

$$S = \int dx_1 \left[\int_0^\infty \frac{1}{2} (\partial_\mu u)^2 dx_2 + V_B(u) \right]. \tag{2.2}$$

In physics $u = u(x)$ is called a real scalar field in euclidean two dimensional space. The boundary potential V_B is in general an arbitrary function of u at the boundary, $x_2 = 0$. For an integrable homogeneous boundary condition V_B is quadratic in u :

$$V_B(u) = \frac{a}{2} u^2. \tag{2.3}$$

Namely it attaches a ‘spring’ with a spring constant a at the boundary.

The action becomes stationary for the solutions of equation of motion

$$\Delta u = \partial_\mu^2 u = (\partial_1^2 + \partial_2^2)u = 0, \tag{2.4}$$

and the boundary condition

$$\frac{\partial}{\partial x_2} u(x_1, 0) = a u(x_1, 0), \quad \text{mixed b.c.} \tag{2.5}$$

It is well known that the solution of the above problem is given by the real and/or imaginary part of holomorphic functions (*(anti) holomorphic map*) superposed appropriately in terms of the image charge method. For the

⁵Throughout this paper the two-dimensional space is euclidean, $x = (x_1, x_2)$ and x_1 is considered as ‘euclidean time’.

limits of $a \rightarrow \infty$ and $a = 0$, the above boundary conditions become the well known

$$\begin{aligned} u(x_1, 0) &= 0, & \text{Dirichlet b.c. } a \rightarrow \infty, \\ \frac{\partial}{\partial x_2} u(x_1, 0) &= 0, & \text{Neumann b.c. } a = 0. \end{aligned}$$

It is very easy to go from harmonic functions to a non-linear sigma model with target space TG . Different sigma models are obtained for different choices of TG . Let

$$u^\alpha, \quad \alpha = 1, \dots, N,$$

be some local coordinates of TG with the metric tensor $g_{\alpha\beta}(u)$. Then a non-linear sigma model (or free field theory taking value in TG) is defined by

$$S = \int dx_1 \left[\int_0^\infty \frac{1}{2} g_{\alpha\beta}(u) \partial_\mu u^\alpha \partial_\mu u^\beta dx_2 + V_B(u) \right]. \quad (2.6)$$

However, there is one big difference. That is the boundary term V_B . In the harmonic function case, the quadratic boundary term (2.3) has a well defined meaning. On an arbitrary manifold, quadratic functions of the local coordinates or other functions do not have an invariant meaning. Therefore in this paper we consider only the *free boundary* case:

$$V_B = 0. \quad (2.7)$$

The equation of motion is

$$\partial_\mu \left(g_{\alpha\beta}(u) \partial_\mu u^\beta(x) \right) = 0, \quad (2.8)$$

and the free boundary condition is

$$g_{\alpha\beta}(u) \partial_2 u^\beta(x) = 0, \quad \text{at the boundary,}$$

or

$$\partial_2 u^\alpha(x_1, 0) = 0. \quad (2.9)$$

In all the explicit examples treated below, the Lagrangians have much simpler forms reflecting the high degrees of symmetry of the systems.

Before closing this section let us remark on the general setting of harmonic maps of manifolds with boundary [21]. Let X and Y be compact riemannian manifolds with boundary. In [21] certain existence theorems of the harmonic maps $f : X \rightarrow Y$ corresponding to the Dirichlet, Neumann and mixed boundary conditions were given ⁶. In all these cases the target manifold Y was assumed to have non-positive riemannian curvature, which was necessary for a heat equation method to work. These results seem to be irrelevant to the non-linear sigma models in this paper, since they all have positive riemannian curvature.

3 Non-Local Charges

In this section we follow the argument of Brézin et al [5] (see also [22]), and derive the infinite set of continuity equations. Let us start by assuming that the ‘gauge’ field

$$A_\mu(x) = A_\mu^{\alpha\beta}(x), \quad \alpha, \beta = 1, 2, \dots, N, \quad \mu = 1, 2$$

satisfy the following two properties for *the solutions of the equation of motion*,

- (i) A_μ is a ‘pure gauge’, namely the corresponding ‘field strength’ vanishes

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0. \quad (3.1)$$

- (ii) A_μ satisfy the ‘continuity equation’

$$\partial_\mu A_\mu(x) = 0. \quad (3.2)$$

In all three examples discussed below, the ‘pure gauge’ condition is trivially satisfied and the ‘continuity equation’ has the dynamical contents. Whereas

⁶Here the boundary conditions were not derived from variations.

in the pseudodual chiral models [4, 22], the ‘continuity equation’ is satisfied trivially and the ‘pure gauge’ condition has the non-trivial dynamical meaning.

In either case, the above two properties (i) and (ii) are encoded neatly into the following ‘linear scattering’ problem

$$\partial_+ \psi^\alpha = - \sum_\beta \frac{A_+^{\alpha\beta}}{1+\lambda} \psi^\beta, \quad (3.3)$$

$$\partial_- \psi^\alpha = - \sum_\beta \frac{A_-^{\alpha\beta}}{1-\lambda} \psi^\beta, \quad (3.4)$$

in which λ is the ‘spectral parameter’ and x_\pm are the two-dimensional complex coordinates

$$x_\pm = x_1 \pm ix_2, \quad \partial_\pm = \frac{\partial}{\partial x_\pm}, \quad (3.5)$$

and

$$A_\pm = \frac{1}{2}(A_1 \mp iA_2).$$

In other words, the two conditions (i) and (ii) can be written as ‘one parameter family of zero-curvature conditions’

$$d\mathcal{A} - \mathcal{A}^2 = 0, \quad \mathcal{A} \equiv -\frac{A_+}{1+\lambda} dx_+ - \frac{A_-}{1-\lambda} dx_-, \quad (3.6)$$

which represents the integrability of (3.3) and (3.4) expressed in terms of a ‘one parameter family of one form’ \mathcal{A} ,

$$d\psi = \mathcal{A}\psi. \quad (3.7)$$

Namely, the the two conditions (i) and (ii) which will be used below to derive the infinite set of ‘continuity equations’ are equivalent with the ‘Lax pair’ (or the ‘linear scattering’ problem) formulation of the non-linear sigma models.

If we define the ‘covariant derivative’ operator

$$D_\mu^{\alpha\beta} = \delta^{\alpha\beta} \partial_\mu + A_\mu^{\alpha\beta}(x),$$

then (3.1) can be expressed as

$$[D_\mu, D_\nu] = 0. \quad (3.8)$$

Then (3.2) can be rewritten as an operator identity:

$$\partial_\mu D_\mu = D_\mu \partial_\mu. \quad (3.9)$$

Based on (3.8) and (3.9), Brézin et al [5] showed the existence of an infinite number of continuity equations inductively. Suppose $J_\mu^{(n)}$ is the n -th conserved current,

$$\partial_\mu J_\mu^{(n)}(x) = 0,$$

then there exists a function $\chi^{(n)}(x)$, such that

$$J_\mu^{(n)}(x) = \epsilon_{\mu\nu} \partial_\nu \chi^{(n)}(x), \quad n \geq 1. \quad (3.10)$$

Here $\epsilon_{\mu\nu}$ is the two-dimensional anti-symmetric tensor with $\epsilon_{12} = 1$. The next conserved current is defined as

$$J_\mu^{(n+1)}(x) = D_\mu \chi^{(n)}, \quad n \geq 0. \quad (3.11)$$

This definition is consistent if we identify

$$\chi^{(0)}(x) = 1, \quad J_\mu^{(1)}(x) = A_\mu(x). \quad (3.12)$$

It is easy to see that $J_\mu^{(n+1)}(x)$ satisfies the continuity equation

$$\begin{aligned} \partial_\mu J_\mu^{(n+1)}(x) &= \partial_\mu D_\mu \chi^{(n)} = D_\mu \partial_\mu \chi^{(n)} = -\epsilon_{\mu\nu} D_\mu J_\nu^{(n)} \\ &= -\epsilon_{\mu\nu} D_\mu D_\nu \chi^{(n-1)} = -[D_1, D_2] \chi^{(n-1)} = 0. \end{aligned} \quad (3.13)$$

4 Conserved Quantities

In the previous section we have seen that the infinite set of continuity equations is a consequence of the equation of motion. But a continuity equation

itself does not give a conserved quantity. In order to obtain the conserved quantities from these continuity equations the boundary conditions play an essential role [16, 17]. Here we show that for the free boundary condition on the half plane the infinite set of non-local charges are not conserved in general. The conservation of these non-local charges imposes very severe constraints which eventually reduce the theory ‘trivial’. We will not discuss the involution property of the corresponding non-local conserved charges on the full plane. (See the comments at the end of this section.)

As discussed in the section two we concentrate on the free boundary,

$$V_B = 0.$$

In this case the non-local charges do not get any contributions from the ‘boundary term’ and are given simply by the integral of the currents. Thus the first non-local charge is given by

$$I^{(1)}(x_1) = \int_0^\infty J_1^{(1)}(x_1, x_2) dx_2. \quad (4.1)$$

Let us check if it is really conserved or not.

$$\frac{d}{dx_1} I^{(1)}(x_1) = \int_0^\infty \frac{\partial}{\partial x_1} J_1^{(1)} dx_2 = - \int_0^\infty \frac{\partial}{\partial x_2} J_2^{(1)} dx_2 = J_2^{(1)}(x_1, 0). \quad (4.2)$$

Here we have used the continuity equation (3.2) and assumed that the currents vanish at $x_2 = \infty$:

$$\lim_{x_2 \rightarrow \infty} J_1^{(1)}(x_1, x_2) = 0 = \lim_{x_2 \rightarrow \infty} J_2^{(1)}(x_1, x_2).$$

Therefore the conservation of the first charge $I^{(1)}(x_1)$ (i.e. independent of x_1) is only achieved when the condition

$$J_2^{(1)}(x_1, 0) = 0 \quad (4.3)$$

is satisfied ⁷. Namely, the current flowing to the endpoint must be vanishing.

⁷ As we see in the explicit examples in sections 5,6,7 this condition is usually equivalent with the free boundary condition on the field.

Likewise the n -th non-local charge is given by the integral of the n -th current $J_\mu^{(n)}$,

$$I^{(n)}(x_1) = \int_0^\infty J_1^{(n)}(x_1, x_2) dx_2.$$

Since the above argument applies equally well to the n -th current $J_\mu^{(n)}$, an additional condition

$$J_2^{(n)}(x_1, 0) = 0 \quad (4.4)$$

must be met in order for it to be conserved.

Let us look into details. Let us start with the second current $J_\mu^{(2)}$. A scalar ‘potential’ $\chi^{(1)}$ is defined by

$$J_1^{(1)} = \partial_2 \chi^{(1)}, \quad J_2^{(1)} = -\partial_1 \chi^{(1)}. \quad (4.5)$$

Then the second current is defined by $J_\mu^{(2)} = D_\mu \chi^{(1)}$. To be more specific

$$J_1^{(2)} = \partial_1 \chi^{(1)} + A_1 \chi^{(1)} = -J_2^{(1)} + J_1^{(1)} \chi^{(1)}, \quad (4.6)$$

$$J_2^{(2)} = \partial_2 \chi^{(1)} + A_2 \chi^{(1)} = J_1^{(1)} + J_2^{(1)} \chi^{(1)}. \quad (4.7)$$

Thus in order that $I^{(2)}$ is conserved

$$J_2^{(2)}(x_1, 0) = J_1^{(1)}(x_1, 0) + J_2^{(1)}(x_1, 0) \chi^{(1)}(x_1, 0) = 0 \quad (4.8)$$

is necessary. Since $A_2(x_1, 0) = J_2^{(1)}(x_1, 0) = 0$, the above condition is equivalent to

$$J_1^{(1)}(x_1, 0) = 0 = A_1(x_1, 0). \quad (4.9)$$

From (4.6) this in turn implies

$$J_1^{(2)}(x_1, 0) = 0. \quad (4.10)$$

By induction we can show that if the first and second non-local charges are conserved then the higher non-local charges $I^{(1)}, \dots, I^{(n)}$ are also conserved

and that this also implies that both $(\mu = 1, 2)$ components of the higher currents vanish at the boundary

$$J_1^{(k)}(x_1, 0) = 0 = J_2^{(k)}(x_1, 0), \quad k = 1, \dots, n. \quad (4.11)$$

The next currents are expressed as

$$\begin{aligned} J_1^{(n+1)} &= \partial_1 \chi^{(n)} + A_1 \chi^{(n)} = -J_2^{(n)} + A_1 \chi^{(n)}, \\ J_2^{(n+1)} &= \partial_2 \chi^{(n)} + A_2 \chi^{(n)} = J_1^{(n)} + A_2 \chi^{(n)}. \end{aligned}$$

Thus by the assumption of the induction we find that the next charge is also conserved and

$$\begin{aligned} J_1^{(n+1)}(x_1, 0) &= -J_2^{(n)}(x_1, 0) + A_1(x_1, 0) \chi^{(n)}(x_1, 0) = 0, \\ J_2^{(n+1)}(x_1, 0) &= J_1^{(n)}(x_1, 0) + A_2(x_1, 0) \chi^{(n)}(x_1, 0) = 0. \end{aligned} \quad (4.12)$$

Therefore the conservation of non-local charges on a half plane imposes very severe constraints.

As a small digression, let us consider the case that $I^{(1)}$ and $I^{(3)}$ are conserved but that the conservation of $I^{(2)}$ is not required. In this case

$$J_1^{(3)} = -J_2^{(2)} + A_1 \chi^{(2)}, \quad (4.13)$$

$$J_2^{(3)} = J_1^{(2)} + A_2 \chi^{(2)}. \quad (4.14)$$

Thus requiring $J_2^{(3)}(x_1, 0) = 0$ means

$$J_1^{(2)}(x_1, 0) = 0, \quad (4.15)$$

which in turn (by (4.6)) means (by using $J_2^{(1)}(x_1, 0) = 0$)

$$A_1(x_1, 0) = J_1^{(1)}(x_1, 0) = 0 \quad (4.16)$$

and $I^{(2)}$ is also conserved.

It should be remarked that throughout the above discussion we used the fact that

$$\chi^{(n)}(x_1, x_2) \quad (4.17)$$

is non-singular at the boundary $x_2 = 0$.

It is worthwhile to comment on the relationship with other approaches to the conserved charges. On the full plane there are many work on how to extract various sets of infinite conserved charges from the ‘one parameter family of zero-curvature’ equations (3.6), [23, 22], which roughly correspond to the various choices of the expansion points, $\lambda = \lambda_0$. Namely, the variety of infinite sets of conserved charges can be understood as different explicit expressions of the ‘one parameter family of conserved charges’.

5 $O(N)$ Sigma Model

Next let us consider the consequences of the above severe constraints coming from the conservation of non-local charges for explicit field theories. The first example is the well known $O(N)$ sigma model which consists of an N -component real scalar field

$$\phi = \{\phi^\alpha\}, \quad \alpha = 1, \dots, N,$$

satisfying the condition

$$\phi \cdot \phi = \sum_{\alpha=1}^N (\phi^\alpha)^2 = 1. \quad (5.1)$$

Namely the target space is the $N - 1$ dimensional unit sphere $TG = S^{N-1}$ in the N dimensional euclidean space. The model is defined by the action

$$S = \frac{1}{2} \int dx_1 \int_0^\infty dx_2 \sum_{\alpha=1}^N (\partial_\mu \phi^\alpha)^2. \quad (5.2)$$

Since we mainly discuss the classical theory the coupling constant is irrelevant. From the stationarity of the action we obtain the equation of motion

$$\partial_\mu^2 \phi + \phi(\partial_\mu \phi \cdot \partial_\mu \phi) = 0, \quad (5.3)$$

and the free boundary condition

$$\partial_2 \phi(x_1, 0) = 0. \quad (5.4)$$

The first current is given by

$$J_\mu^{(1)} = A_\mu = A_\mu^{\alpha\beta} = 2 \left(\phi^\alpha \partial_\mu \phi^\beta - (\partial_\mu \phi^\alpha) \phi^\beta \right). \quad (5.5)$$

Thus the conservation of the first charge is a consequence of the free boundary condition. It is straightforward to verify the continuity equation for A_μ

$$\partial_\mu A_\mu^{\alpha\beta} = 0, \quad (5.6)$$

by using the equation of motion (5.3). It is also straightforward to show that the current A_μ is a ‘pure gauge’:

$$\partial_1 A_2 - \partial_2 A_1 + [A_1, A_2] = 0.$$

The conservation of the first and second charges is equivalent with the condition

$$A_\mu^{\alpha\beta}(x_1, 0) = 0, \quad \mu = 1, 2. \quad (5.7)$$

By multiplying ϕ^α we obtain

$$\partial_\mu \phi^\beta - (\phi^\alpha \cdot \partial_\mu \phi^\alpha) \phi^\beta = 0. \quad (5.8)$$

Since $\phi \cdot \phi = 1$ implies $\phi^\alpha \cdot \partial_\mu \phi^\alpha = 0$ at any point including the boundary, we obtain

$$\partial_\mu \phi^\beta = 0, \quad \text{at the boundary,} \quad \beta = 1, \dots, N,$$

or

$$\partial_1 \phi^\beta(x_1, 0) = 0 = \partial_2 \phi^\beta(x_1, 0). \quad (5.9)$$

The first equation implies

$$\partial_1^n \phi^\beta(x_1, 0) = 0, \quad n \geq 1. \quad (5.10)$$

It is now obvious that the second charge is not conserved in general and its conservation requires very strong conditions, which allow only trivial solutions as we will see below.

Next let us calculate

$$\partial_2^2 \phi(x_1, 0)$$

by using the equation of motion at small x_2 :

$$\begin{aligned} \partial_2^2 \phi(x_1, x_2) &= (\partial_1^2 + \partial_2^2 - \partial_1^2) \phi(x_1, x_2) \\ &= -\phi(x_1, x_2)(\partial_\mu \phi \cdot \partial_\mu \phi)(x_1, x_2) - \partial_1^2 \phi(x_1, x_2). \end{aligned} \quad (5.11)$$

By taking the limit $x_2 \rightarrow 0$ in the above expression and using (5.9), (5.10), we obtain

$$\partial_2^2 \phi(x_1, 0) = 0. \quad (5.12)$$

Similarly, by further differentiation of (5.11) we obtain

$$\begin{aligned} \partial_2^3 \phi(x_1, x_2) &= (\partial_2 \phi(x_1, x_2))(\partial_\mu \phi \cdot \partial_\mu \phi)(x_1, x_2) \\ &+ \phi(x_1, x_2) \partial_2 (\partial_\mu \phi \cdot \partial_\mu \phi)(x_1, x_2) - \partial_1^2 \partial_2 \phi(x_1, x_2). \end{aligned} \quad (5.13)$$

Again by using (5.9) and (5.10), we obtain

$$\partial_2^3 \phi(x_1, 0) = 0. \quad (5.14)$$

By repeating the same argument, we arrive at

$$\partial_2^n \phi(x_1, 0) = 0, \quad n \geq 1, \quad (5.15)$$

which together with (5.10) would exclude any non-trivial solutions with real analytic dependence on x_2 .

6 Principal Chiral Model

The next example is the principal chiral model, that is the sigma model taking value in a group manifold. It is defined by the action

$$S = \int dx_1 \int_0^\infty dx_2 \text{Tr} \left(\partial_\mu g \partial_\mu g^{-1} \right), \quad (6.1)$$

in which $g = g(x)$ takes value in a group of $N \times N$ matrices. The lowest member of the current is given by

$$A_\mu = g^{-1} \partial_\mu g, \quad (6.2)$$

which is obviously a pure gauge:

$$\partial_1 A_2 - \partial_2 A_1 + [A_1, A_2] = 0.$$

The equation of motion reads

$$\partial_\mu A_\mu = \partial_\mu (g^{-1} \partial_\mu g) = 0 \quad (6.3)$$

and the free boundary condition is given by

$$(g^{-1} \partial_2 g)(x_1, 0) = 0. \quad (6.4)$$

Therefore the conservation of the first charge is a consequence of the free boundary condition, too. The condition for conservation of the first and second charges can be written succinctly

$$g^{-1} \partial_\mu g = 0, \quad \text{at } x_2 = 0$$

or

$$\partial_\mu g(x_1, 0) = 0, \quad \mu = 1, 2. \quad (6.5)$$

Again the conservation of the second charge is not guaranteed in general.

We will show below that the solutions of the equation of motion preserving the infinite set of non-local charges are trivial. From the equation of motion, we obtain

$$(\partial_1^2 g + \partial_2^2 g) = (\partial_\mu g) g^{-1} (\partial_\mu g). \quad (6.6)$$

By going to the boundary ($x_2 \rightarrow 0$) and by using (6.5) we obtain

$$\partial_2^2 g(x_1, 0) = 0. \quad (6.7)$$

And from (6.5) we find also

$$\partial_1^n g(x_1, 0) = 0, \quad n \geq 1. \quad (6.8)$$

By repeating similar arguments we can show

$$\partial_2^n g(x_1, 0) = 0, \quad n \geq 1. \quad (6.9)$$

As in the case of the $O(N)$ sigma model this would exclude any non-trivial solutions with real analytic dependence on x_2 .

7 \mathbb{CP}^{N-1} and Grassmannian Sigma Models

There are many ways to express the \mathbb{CP}^{N-1} and Grassmannian sigma models. The complex Grassmannian manifold $Gr(N, m)$ is a space of m -frames in the complex N -dimensional vector space C^N . Let X be an m -frame (an $N \times m$ matrix)

$$X = (e_{j_1}, \dots, e_{j_m}),$$

in which $\{e_j\}$, $j = 1, \dots, N$ is an orthonormal basis of C^N :

$$e_j^\dagger \cdot e_k = \delta_{jk}.$$

Then X satisfies the constraint

$$X^\dagger X = 1_m, \quad m \times m \text{ unit matrix.} \quad (7.1)$$

We choose as a field variable of the $Gr(N, m)$ model a projector $P = P(x)$, which is an $N \times N$ matrix

$$P = P(x) = X(x)X(x)^\dagger, \quad (7.2)$$

Then it is obvious that P has the properties of the projector:

$$P^2 = P, \quad P^\dagger = P.$$

The special case of $m = 1$ corresponds to the CP^{N-1} model.

The first current is defined by

$$A_\mu = 2[P, \partial_\mu P], \quad (7.3)$$

and the equation of motion reads

$$0 = \partial_\mu A_\mu = 2[P, \partial_\mu^2 P]. \quad (7.4)$$

They can be obtained by embedding (reduction) the $Gr(N, m)$ sigma model into the principal chiral model :

$$g = 1 - 2P. \quad (7.5)$$

This has the properties

$$\begin{aligned} g^2 &= (1 - 2P)^2 = 1 - 4P + 4P^2 = 1, \\ g^\dagger &= 1 - 2P^\dagger = g. \end{aligned}$$

It is easy to derive (7.3) and (7.4) from (6.2) and (6.3), respectively. The free boundary condition (6.4) is rewritten as

$$\partial_2 P(x_1, 0) = 0. \quad (7.6)$$

As in the $O(N)$ and the principal chiral models the second charge is not conserved in general, unless an additional condition

$$\partial_1 P(x_1, 0) = 0, \quad (7.7)$$

is satisfied. If we require that the first and second charges to be conserved, then the solutions satisfy the following constraints

$$\partial_1^n P(x_1, 0) = 0 = \partial_2^n P(x_1, 0), \quad n \geq 1, \quad (7.8)$$

which are derived from (6.8), (6.9) and $g = 1 - 2P$ relation. As in the case of the $O(N)$ and principal chiral sigma model this would exclude any non-trivial solutions with real analytic dependence on x_2 .

It is instructive to see that the simplest solutions of the CP^{N-1} and Grassmannian sigma models, the (anti) instanton solutions (holomorphic maps) do not satisfy the free boundary condition (7.6), unless it is a trivial (constant) solution. Let us consider the $Gr(N, m)$ model. The general instanton solution is given by [6]

$$X = F(F^\dagger F)^{-1/2}, \quad P = F(F^\dagger F)^{-1}F^\dagger, \quad (7.9)$$

in which F is an $N \times m$ matrix consisting of linearly independent holomorphic vectors f_1, \dots, f_m , which can be chosen arbitrarily,

$$F = F(x_+) = (f_1(x_+), \dots, f_m(x_+)), \quad \partial_- F = 0. \quad (7.10)$$

Here use is made of the complex two dimensional coordinates $x_\pm = x_1 \pm ix_2$. It is easy to see

$$\partial_+ P = (\mathcal{D}_+ F)(F^\dagger F)^{-1}F^\dagger, \quad \text{and} \quad \partial_- P = F(F^\dagger F)^{-1}(\mathcal{D}_+ F)^\dagger, \quad (7.11)$$

where the ‘covariant derivative’ operator \mathcal{D}_+ is defined by

$$\mathcal{D}_+ F = \partial_+ F - F(F^\dagger F)^{-1}F^\dagger \partial_+ F = (1 - P)\partial_+ F.$$

From (7.11) and the properties of the projection operator it is obvious

$$P(\partial_+ P) = 0 = (\partial_- P)P. \quad (7.12)$$

By making a combination $\partial_- [P(\partial_+ P)] - \partial_+ [(\partial_- P)P]$, we easily find that P satisfies the equation of motion

$$[P, \partial_+ \partial_- P] = 0.$$

For more general solutions, see [9].

Now let us consider the boundary condition of the instanton solution. The free boundary condition (7.6) is rewritten as

$$i(\partial_+ - \partial_-)P(x_1, 0) = 0,$$

or by using (7.11)

$$\left((\mathcal{D}_+ F)(F^\dagger F)^{-1} F^\dagger - F(F^\dagger F)^{-1} (\mathcal{D}_+ F)^\dagger\right) = 0, \quad (7.13)$$

It is easy to see $F^\dagger \mathcal{D}_+ F = 0$. Multiplying F from the right to (7.13), we obtain

$$(\mathcal{D}_+ F)(x_1, 0) = 0. \quad (7.14)$$

This in turn means

$$\partial_+ P(x_1, 0) = 0 = \partial_- P(x_1, 0).$$

Therefore all the holomorphic vectors f_1, \dots, f_m are constants and the instanton solution of the $Gr(N, m)$ model satisfying the free boundary condition is trivial. It should be noted that the conservation of the second charge is not imposed here.

8 Summary and Comments

We have shown that at the classical level the integrable non-linear sigma models, for example, the $O(N)$, the principal chiral, the CP^{N-1} and the complex Grassmannian sigma models lose the integrability when restricted on a half plane with free boundary. Because of the conformal invariance of the equation of motion and the Riemann mapping theorem, the result is valid in any simply connected domain in two dimensional space with free boundary. This result is in sharp contrast with other integrable models, sine-Gordon, non-linear Schrödinger and affine Toda field theories which are known to be integrable with various boundary conditions including the free boundary.

However, the situation of the non-linear sigma models might be a rule rather than an exception. Let us consider a well known integrable field theory on the whole line: the KdV equation

$$u_t = u_{xxx} - 6uu_x, \quad u = u(t, x), \quad u_t = \partial_t u, \quad u_x = \partial_x u, \quad \text{etc.} \quad (8.1)$$

If we restrict it to a half line $0 \leq x < \infty$, the first and second charges $I^{(1)}$ and $I^{(2)}$

$$I^{(1)} = \int_0^\infty u \, dx, \quad I^{(2)} = \frac{1}{2} \int_0^\infty u^2 \, dx,$$

are not conserved for either of the boundary conditions:

$$u(t, 0) = 0, \quad \text{or} \quad u_x(t, 0) = 0.$$

It is easy to see

$$\begin{aligned} \frac{d}{dt} I^{(1)}(t) &= -[u_{xx} - 3u^2] \Big|_{x=0} \neq 0, \\ \frac{d}{dt} I^{(2)}(t) &= -[uu_{xx} - \frac{1}{2}u_x^2 - 2u^3] \Big|_{x=0} \neq 0. \end{aligned}$$

It should be noted that the KdV equation does not have a Lagrangian formulation. It is easy to find some other integrable equations which lose integrability when restricted to a half line.

Here we would like to comment on the well known correspondence (reduction) of the $O(3)$ sigma model and the sine-Gordon theory on the full plane established by Pohlmeyer [24]. This correspondence, if true on the half plane, would lead to a contradiction, since the $O(3)$ sigma model with a free boundary is not integrable but, as mentioned several times, the sine-Gordon theory on the half plane is integrable with the free and the other boundary interactions. The correspondence is achieved by utilising the full conformal invariance of the theory in the two-dimensional Minkowski space;

$$\xi \rightarrow \xi' = f(\xi), \quad \eta \rightarrow \eta' = g(\eta), \quad \xi = (t + x)/2, \quad \eta = (t - x)/2, \quad (8.2)$$

where f and g are arbitrary (the ‘left’ and ‘right’ transformation) functions. By appropriate choice of the functions f and g one can transform the $O(3)$ sigma model variable ϕ , such that it satisfies

$$(\phi)^2 = (\partial_\xi \phi)^2 = (\partial_\eta \phi)^2 = 1. \quad (8.3)$$

Thus the only remaining degree of freedom is an ‘angle’ Ψ

$$\cos \Psi = \partial_\xi \phi \cdot \partial_\eta \phi, \quad (8.4)$$

which should satisfy the sine-Gordon equation

$$\partial_\xi \partial_\eta \Psi + \sin \Psi = 0. \quad (8.5)$$

On the half plane, on the other hand, the conformal transformation depends on only one arbitrary function h ,

$$\xi' = h(\xi), \quad \eta' = h(\eta), \quad (8.6)$$

which is not sufficient to reduce the system to the form (8.3). Thus the $O(3)$ sigma model cannot be reduced to the sine-Gordon theory on the half plane.

It is very challenging to consider the *quantum integrability* of non-linear sigma models on the half plane. As mentioned in section 1 the Reflection equation of these models offers interesting extensions of the Yang-Baxter equations and its related algebras for the known exact factorisable S-matrices. It should be remarked that the relationship between the classical and quantum theories of non-linear sigma models is not so straightforward as that of sine-Gordon and/or affine Toda field theories. At the classical level the non-linear sigma models are conformally invariant, i.e., massless, whereas the quantum spectrum consists of massive particles belonging to certain representations of appropriate Lie algebras.

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